Log Linearisation as a Coordinate Representation of Taylor Expansion

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Abstract

Macroeconomists approximate nonlinear models using log linearisation, while mathematics and the physical sciences rely on Taylor expansions in levels. This note shows that the distinction is purely representational: log linearisation is a particular coordinate expression of a Taylor expansion. Separating the order of approximation from the choice of deviation metric, I derive explicit second-order mappings between log and per-unit deviations and show that, when treated consistently, they can both be used to approximate economic equilibrium conditions. The analysis also clarifies why log deviations preserve multiplicative relations exactly, whereas per-unit deviations preserve additive identities, providing a unified foundation for common approximation practices in macroeconomics.

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1 Introduction

Log linearisation is a standard tool in macroeconomics. It is routinely used to study the local properties of nonlinear dynamic models, to characterise equilibrium dynamics, and to obtain tractable representations of policy rules. Canonical treatments include early methodological contributions such as Uhlig (2001), as well as textbook expositions and widespread applications in modern DSGE analysis (Woodford, 2003; Galí, 2015).

At the same time, higher-order (and in particular second-order) approximation methods have become central to quantitative macroeconomics, particularly for welfare analysis and policy evaluation. Influential contributions include Schmitt-Grohé and Uribe (2004), who develop second-order perturbation methods for rational expectations models, and Benigno and Woodford (2012), who derive linear–quadratic approximations to optimal policy problems from nonlinear foundations.

Because of this extensive use, log linearisation is often presented and taught to economics students as *the* method to approximate models in a tractable form. However, any student of mathematics would be troubled by this statement: the approach to approximation taught in undergraduate mathematics classes is Taylor approximation, in which log deviations never appear, and which constitutes one of the backbones of modern calculus. How do these two approaches relate?

This note offers such a clarification. The central idea is to separate two conceptually distinct steps that are often conflated in economics practice: (i) the Taylor expansion of equations in levels around a reference point, which determines the approximation order; and (ii) the choice of deviation metric used to express the resulting approximation. Log deviations turn out to be only one of the possible choices of coordinates to approximate models. An alternative is constituted by "per-unit" deviations, which are shown to be related to log deviations by explicit second-order mappings, implying that both representations describe the same local approximation when handled consistently.

Relatedly, numerous different algebraic approaches to log linearisation have been introduced. This note, by developing a coherent treatment of the relationship between log linearisation and Taylor approximation, can contribute to the unification of these a approaches which can be shown to be equivalent.

Two further results explain common modelling practice. First, additive identities are preserved exactly in per-unit deviations but not in log deviations. Second, multiplicative relationships become exactly linear identities in log coordinates but not in per-unit form. These properties explain why log linearisation is algebraically convenient for production functions, Euler equations, and pricing relationships, while per-unit representations are natural for accounting identities. Log linearisation then arises as the natural choice in models in which multiplicative relations dominate.

By making these relationships explicit, the note seeks to provide a clearer foundation for first- and second-order approximations and to clarify common but often under-articulated practices in applied macroeconomic analysis.

The motivation for this note arose while working with second-order approximations in macroeconomic models, where these distinctions play an implicit but important role (Drechsel et al., 2025).

2 Notation and Deviation Measures

Let $X \in \mathbb{R}$ denote a scalar variable with reference value $\bar{X}$. Throughout the note, we study local approximations of equations around $\bar{X}$. Unless otherwise stated, it is assumed that $\bar{X} \neq 0$. When log deviations are used, we additionally require $\bar{X} > 0$.

Definition 1 (Per-unit deviation) *The per-unit deviation of X from $\bar{X}$ is defined as*

$$\tilde{x} \equiv \frac{X - \bar{X}}{\bar{X}}.$$

Per-unit deviations measure proportional changes relative to the reference value. They are well defined whenever $\bar{X} \neq 0$ and do not require X to be strictly positive. In particular, per-unit deviations are naturally suited to equations involving sums or adding-up constraints, since additive relationships are preserved exactly in per-unit form.

Throughout the note, the notation $\tilde{x}$ is used exclusively to denote per-unit deviations.

Definition 2 (Log deviation) *When $X > 0$, the log deviation of X from $\bar{X}$ is defined as*

$$\hat{x} \equiv \log X - \log \bar{X}.$$

Log deviations measure changes in logarithms and are invariant to rescaling of units. They are particularly convenient in environments with multiplicative relationships, as products and powers translate into sums and coefficients in log form.

The notation $\hat{x}$ is used throughout to denote log deviations.

All approximations considered in this note are local to the reference point $\bar{X}$. Their accuracy depends on the magnitude of deviations from $\bar{X}$ and on the curvature of the underlying functions evaluated at that point. In particular, neither log deviations nor per-unit deviations are globally scale free; both define local coordinate systems around

$\bar{X}$. The choice of reference point is typically dictated by the steady state of a model, but the analysis applies to any interior point at which the relevant derivatives exist.

For clarity, the following terminology is used consistently throughout the note:

- *Linearisation* refers to a first-order Taylor approximation.
- *Quadratic approximation* refers to a second-order Taylor approximation.
- *Log linearisation* and *per-unit linearisation* refer to expressing the resulting approximation in terms of log deviations or per-unit deviations, respectively.

This distinction emphasises that the order of approximation is determined by the Taylor expansion in levels, while the choice between log and per-unit deviations is a matter of representation.

3 Mapping Between Log and Per-Unit Deviations

This section establishes the formal relationship between log deviations and per-unit deviations. The results clarify how first- and second-order approximations expressed in one deviation metric translate into the other and show that remainder terms are preserved under these transformations.

Lemma 1 (Log and per-unit deviations) *For X in a neighborhood of $\bar{X}$,*

$$\tilde{x} = \hat{x} + \tfrac{1}{2}\hat{x}^2 + o(\hat{x}^2), \tag{1}$$

and equivalently,

$$\hat{x} = \tilde{x} - \tfrac{1}{2}\tilde{x}^2 + o(\tilde{x}^2). \tag{2}$$

Proof. By definition,

$$X = \bar{X}e^{\hat{x}},$$

so that

$$\tilde{x} = e^{\hat{x}} - 1.$$

A second-order Taylor expansion of $e^{\hat{x}}$ around zero yields

$$e^{\hat{x}} = 1 + \hat{x} + \tfrac{1}{2}\hat{x}^2 + o(\hat{x}^2),$$

which implies (1).

Conversely, writing

$$\hat{x} = \log(1 + \tilde{x}),$$

a second-order Taylor expansion of $\log(1 + u)$ around $u = 0$ gives

$$\log(1 + \tilde{x}) = \tilde{x} - \tfrac{1}{2}\tilde{x}^2 + o(\tilde{x}^2),$$

which implies (2). ■

The next result shows that the notion of second-order accuracy is invariant to the choice of deviation metric.

Lemma 2 (Equivalence of small-o classes) *When $X \to \bar{X}$*

$$\frac{\hat{x}}{\tilde{x}} \to 1 \, .$$

Consequently, for any $n > 0$,

$$o(\hat{x}^{\,n}) = o(\tilde{x}^{\,n}),$$

where equality is understood as equality of sets of functions under the change of variables between $\hat{x}$ and $\tilde{x}$.

Proof. We can directly show

$$\lim_{X \to \bar{X}} \frac{\hat{x}}{\tilde{x}} = \lim_{X \to \bar{X}} \frac{\log X - \log \bar{X}}{\frac{X - \bar{X}}{\bar{X}}} = \lim_{Y \to 1} \frac{\log Y}{Y - 1} \, , \tag{3}$$

which is well known to equal 1. Since $\tilde{x}/\hat{x} \to 1$, it follows that $(\tilde{x}/\hat{x})^n \to 1$ for any $n > 0$.

Let $f \in o(\hat{x}^n)$. Then

$$\frac{f(\hat{x})}{\tilde{x}^n} = \frac{f(\hat{x})}{\hat{x}^n} \left(\frac{\hat{x}}{\tilde{x}}\right)^n \longrightarrow 0 \quad \text{as} \;\; \hat{x} \to 0 \, .$$

so $f \in o(\tilde{x}^n)$.

The reverse inclusion follows by the same argument. ■

Lemma 2 implies that a second-order approximation derived using per-unit deviations is also second-order accurate when expressed in log deviations, and vice versa. As a result, the choice between log and per-unit representations affects the coefficients of quadratic terms but not the order of the approximation. The result generalises to any order of approximation.

As a corollary, the two lemmata above imply that log and per-unit deviations are equivalent to the first order, as summarised in the following corollary.

Corollary 3 *Log and per-unit deviations coincide to the first order,*

$$\tilde{x} = \hat{x} + o(\hat{x}) , \qquad \hat{x} = \tilde{x} + o(\tilde{x}) . \tag{4}$$

4 A Unified Approximation Procedure

This section presents a systematic procedure for deriving first- and second-order approximations of economic equations. The procedure separates the Taylor expansion in levels from the choice of deviation metric and applies equally to log and per-unit representations.

Let

$$F(X_1, \ldots, X_n) = 0$$

be a scalar equation, where each X_i has reference value $\bar{X}_i$. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\bar{\mathbf{X}} = (\bar{X}_1, \ldots, \bar{X}_n)$. Assume that F is twice continuously differentiable in a neighbourhood of $\bar{\mathbf{X}}$.

The goal is to obtain a first- or second-order approximation of $F(\mathbf{X}) = 0$ around $\bar{\mathbf{X}}$ and to express this approximation in terms of either per-unit deviations or log deviations.

4.1 Step 1: Taylor expansion in levels

A second-order Taylor expansion of F around $\bar{\mathbf{X}}$ yields

$$\begin{aligned} 0 = F(\bar{\mathbf{X}}) + \sum_{i=1}^{n} F_i(\bar{\mathbf{X}})(X_i - \bar{X}_i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij}(\bar{\mathbf{X}})(X_i - \bar{X}_i)(X_j - \bar{X}_j) \\ + o\big(\|\mathbf{X} - \bar{\mathbf{X}}\|^2\big) , \end{aligned} \tag{5}$$

where F_i and F_{ij} denote first and second partial derivatives.

4.2 Step 2: Express the expansion in per-unit deviations

For each i with $\bar{X}_i \neq 0$, define the per-unit deviation

$$\tilde{x}_i \equiv \frac{X_i - \bar{X}_i}{\bar{X}_i}, \qquad \text{so that} \qquad X_i - \bar{X}_i = \bar{X}_i \, \tilde{x}_i.$$

Substituting into (5) yields

$$0 = F(\bar{\mathbf{X}}) + \sum_{i=1}^{n} F_i(\bar{\mathbf{X}})\,\bar{X}_i\,\tilde{x}_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij}(\bar{\mathbf{X}})\,\bar{X}_i\bar{X}_j\,\tilde{x}_i\tilde{x}_j + o\big(\|\tilde{\mathbf{x}}\|^2\big)\,, \tag{6}$$

where $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)$.

A first-order approximation is obtained by dropping all quadratic and higher-order terms in (6).

4.3 Step 3: Convert to log deviations (if desired)

When $X_i > 0$ and $\bar{X}_i > 0$, define the log deviation $\hat{x}_i \equiv \log X_i - \log \bar{X}_i$. Lemma 1 gives the second-order relation

$$\tilde{x}_i = \hat{x}_i + \tfrac{1}{2}\hat{x}_i^2 + o(\hat{x}_i^2).$$

Using Lemma 2 to preserve the remainder order under the change of variables, substituting into (6) yields

$$\begin{aligned} 0 = F(\bar{\mathbf{X}}) &+ \sum_{i=1}^{n} F_i(\bar{\mathbf{X}})\,\bar{X}_i\,\hat{x}_i + \frac{1}{2}\sum_{i=1}^{n} F_i(\bar{\mathbf{X}})\,\bar{X}_i\,\hat{x}_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij}(\bar{\mathbf{X}})\,\bar{X}_i\bar{X}_j\,\hat{x}_i\hat{x}_j \\ &+ o\big(\|\hat{\mathbf{x}}\|^2\big)\,, \end{aligned} \tag{7}$$

where $\hat{\mathbf{x}} = (\hat{x}_1, \dots, \hat{x}_n)$.

The additional second-order term $\frac{1}{2}\sum_i F_i(\bar{\mathbf{X}})\bar{X}_i\hat{x}_i^2$ arises purely from the change of coordinates between per-unit and log deviations. Conversely, an approximation initially written in log deviations can be converted into per-unit deviations using the inverse mapping in Lemma 1.

Importantly, the order of approximation is determined by the Taylor expansion in levels. The subsequent choice of deviation metric changes coefficients (and may introduce or remove quadratic terms) but does not change the asymptotic order of the approximation, provided the coordinate change is carried out at the corresponding order.[1]

[1] For instance, using only $\hat{x}_i = \tilde{x}_i + o(\tilde{x}_i)$ in a second-order conversion discards essential quadratic terms and effectively reduces the conversion to first-order accuracy.

5 Exactness of Linear Representations for Sums and Products

This section clarifies why log and per-unit linear representations are commonly used in practice. The key observation is that certain functional forms are expressed as exactly linear (up to any order of approximation) under one representation but not under the other. In particular, additive identities are preserved exactly in per-unit form, while multiplicative relationships become exactly linear in log form.

5.1 Additive identities

Lemma 4 (Exactness of per-unit deviations for sums) *Let*

$$Y = \sum_{k=1}^{m} X_k$$

with reference values satisfying $\bar{Y} = \sum_{k=1}^{m} \bar{X}_k$ *and* $\bar{Y} \neq 0$*. Define per-unit deviations*

$$\tilde{y} \equiv \frac{Y - \bar{Y}}{\bar{Y}}, \qquad \tilde{x}_k \equiv \frac{X_k - \bar{X}_k}{\bar{X}_k},$$

and let $s_k \equiv \bar{X}_k/\bar{Y}$*. Then the identity implies the exact relationship*

$$\tilde{y} = \sum_{k=1}^{m} s_k \tilde{x}_k. \tag{8}$$

By contrast, expressing an additive identity in log deviations introduces second- and higher-order terms.

Proof. The additive identity implies $Y - \bar{Y} = \sum_{k=1}^{m}(X_k - \bar{X}_k)$. Dividing both sides by $\bar{Y}$ yields

$$\tilde{y} = \frac{Y - \bar{Y}}{\bar{Y}} = \sum_{k=1}^{m} \frac{\bar{X}_k}{\bar{Y}} \frac{X_k - \bar{X}_k}{\bar{X}_k} = \sum_{k=1}^{m} s_k \tilde{x}_k,$$

which establishes (8). The statement for log deviations follows from Lemma 1. ■

Lemma 4 shows that per-unit linearisation is not an approximation for additive identities but an exact rewriting. Log representations, by contrast, introduce curvature through the logarithmic transformation.

5.2 Multiplicative relationships

Lemma 5 (Exactness of log deviations for multiplicative relationships) *Let*

$$Y = C \prod_{k=1}^{m} X_k^{a_k}, \qquad C > 0,\ \bar{Y} > 0,\ \bar{X}_k > 0,$$

with reference values satisfying $\bar{Y} = C \prod_{k=1}^{m} \bar{X}_k^{a_k}$. *Define log deviations*

$$\hat{y} \equiv \log Y - \log \bar{Y}, \qquad \hat{x}_k \equiv \log X_k - \log \bar{X}_k.$$

Then the exact relationship

$$\hat{y} = \sum_{k=1}^{m} a_k \hat{x}_k \tag{9}$$

holds.

By contrast, expressing the same relationship in per-unit deviations yields

$$1 + \tilde{y} = \prod_{k=1}^{m} (1 + \tilde{x}_k)^{a_k},$$

so that the first-order per-unit approximation

$$\tilde{y} = \sum_{k=1}^{m} a_k \tilde{x}_k + o(\|\tilde{\mathbf{x}}\|)$$

is not exact: interaction terms appear at second order.

Proof. Taking logarithms,

$$\log Y = \log C + \sum_{k=1}^{m} a_k \log X_k, \qquad \log \bar{Y} = \log C + \sum_{k=1}^{m} a_k \log \bar{X}_k.$$

Subtracting yields (9). The per-unit expression follows by dividing by $\bar{Y}$ and substituting $X_k/\bar{X}_k = 1 + \tilde{x}_k$. ■

Lemma 5 shows that log linearisation linearises the multiplicative identity, yielding a representation that is exact at first order. Per-unit deviations, by contrast, introduce second-order interaction terms even for simple multiplicative relationships.

5.3 Log linearisation as standard practice

Taken together, Lemmas 4 and 5 explain the widespread use of log linearisation in macroeconomics. Log deviations are convenient not because they provide a higher-order approximation, but because they preserve the linear structure of multiplicative relationships. Per-unit deviations play an analogous role for additive identities.

More generally, log linearisation should be understood as a coordinate choice rather than a distinct approximation method. In models that combine multiplicative equations with adding-up constraints, a mixed representation may be preferable.

6 Examples

This section illustrates the results of Section 4 by working through two canonical examples, an additive accounting identity and a multiplicative Cobb–Douglas technology, and deriving second-order approximations in both per-unit and log deviations. The calculations make explicit how the "wrong" representation introduces second-order terms, while the "right" representation preserves linear structure.

6.1 Additive identity: $Y = C + I + G$

Consider the accounting identity

$$Y = C + I + G, \qquad \bar{Y} = \bar{C} + \bar{I} + \bar{G}, \qquad \bar{Y} \neq 0.$$

Per-unit representation (exact at all orders). Define shares

$$s_C \equiv \frac{\bar{C}}{\bar{Y}}, \qquad s_I \equiv \frac{\bar{I}}{\bar{Y}}, \qquad s_G \equiv \frac{\bar{G}}{\bar{Y}}, \qquad s_C + s_I + s_G = 1.$$

then

$$\tilde{y} = s_C \tilde{c} + s_I \tilde{i} + s_G \tilde{g}, \tag{10}$$

which is exact (and therefore also exact to second and higher orders).

Log representation (second-order approximation). Since

$$\tilde{y} = \hat{y} + \tfrac{1}{2}\hat{y}^2 + o(\hat{y}^2), \quad \tilde{c} = \hat{c} + \tfrac{1}{2}\hat{c}^2 + o(\hat{c}^2), \quad \tilde{i} = \hat{i} + \tfrac{1}{2}\hat{i}^2 + o(\hat{i}^2), \quad \tilde{g} = \hat{g} + \tfrac{1}{2}\hat{g}^2 + o(\hat{g}^2)\ ,$$

substituting (10) yields a second-order log-deviation approximation:

$$\hat{y} + \frac{1}{2}\hat{y}^2 = \left(s_C\hat{c} + s_I\hat{\imath} + s_G\hat{g}\right) + \frac{1}{2}\left(s_C\hat{c}^2 + s_I\hat{\imath}^2 + s_G\hat{g}^2\right) + o(\|\hat{\mathbf{x}}\|^2), \tag{11}$$

where $\hat{\mathbf{x}} = (\hat{c}, \hat{\imath}, \hat{g})$. We can also express the relationship above as

$$\hat{y} = s_C\hat{c} + s_I\hat{\imath} + s_G\hat{g} + \frac{1}{2}\left(s_C(1-s_C)\hat{c}^2 + s_I(1-s_I)\hat{\imath}^2 + s_G(1-s_G)\hat{g}^2\right) + \tag{12}$$

$$-\left(s_C s_I\hat{c}\hat{\imath} + s_C s_G\hat{c}\hat{g} + s_I s_G\hat{\imath}\hat{g}\right) + o(\|\tilde{\mathbf{x}}\|^2), \tag{13}$$

The equation above makes explicit the point of Lemma 4: per-unit deviations preserve the additive identity exactly, whereas log deviations introduce quadratic (and higher-order) terms.

6.2 Multiplicative relationship: Cobb–Douglas

Consider a Cobb–Douglas production function

$$Y = AK^{\alpha}L^{1-\alpha}, \qquad \bar{Y} = \bar{A}\bar{K}^{\alpha}\bar{L}^{1-\alpha}, \qquad \bar{A}, \bar{K}, \bar{L}, \bar{Y} > 0.$$

Define per-unit deviations $\tilde{y}, \tilde{a}, \tilde{k}, \tilde{l}$ and log deviations $\hat{y}, \hat{a}, \hat{k}, \hat{l}$.

Log representation (exact, hence exact to second order). Taking logs and subtracting the steady state yields the identity

$$\hat{y} = \hat{a} + \alpha\hat{k} + (1-\alpha)\hat{l}, \tag{14}$$

which is exact and therefore contains no quadratic terms.

Per-unit representation (second-order approximation). In per-unit terms instead we find

$$\tilde{y} - \frac{1}{2}\tilde{y}^2 = \tilde{a} - \frac{1}{2}\tilde{a}^2 + \alpha\tilde{k} - \frac{1}{2}\alpha\tilde{k}^2 + (1-\alpha)\tilde{l} - \frac{1}{2}(1-\alpha)\tilde{l}^2 + o(\|\tilde{\mathbf{x}}\|^2), \tag{15}$$

Equivalently,

$$\tilde{y} = \tilde{a} + \alpha\tilde{k} + (1-\alpha)\tilde{l} + \alpha\tilde{a}\,\tilde{k} + (1-\alpha)\tilde{a}\,\tilde{l} - \frac{1}{2}\alpha(1-\alpha)\left(\tilde{k}^2 + \tilde{l}^2 - \tilde{k}\tilde{l}\right) + o(\|\tilde{\mathbf{x}}\|^2). \tag{16}$$

Equation (16) makes explicit the point of Lemma 5: the log representation is linear and exact, while the per-unit representation introduces quadratic and interaction terms at second order.

6.3 Summary

For additive identities, the per-unit representation is exact at all orders, while the log representation introduces curvature through $\log(1+\cdot)$. For multiplicative relationships, the log representation is exact, while the per-unit representation introduces quadratic and interaction terms from expanding products and powers. These examples illustrate that the choice between log and per-unit deviations is primarily a choice of algebraic convenience rather than a choice of approximation order.

7 Conclusion

This note has provided a unified treatment of log and per-unit linearisation for first- and second-order approximations of economic equations. By separating the Taylor expansion in levels from the choice of deviation metric, it has shown that log and per-unit representations are best understood as alternative coordinate systems rather than distinct approximation methods.

Two key results clarify this perspective. First, log deviations and per-unit deviations are related by explicit second-order mappings, implying that the order of an approximation is preserved under a change of representation when the mapping is applied at the corresponding order. Second, certain functional forms are expressed as exactly linear under one representation but not the other: additive identities are exact in per-unit form, while multiplicative relationships are exact in log form.

The main implication is practical rather than methodological. Log linearisation is not inherently more accurate than per-unit linearisation; its usefulness derives from the prevalence of multiplicative relationships in economic models. In settings dominated by adding-up constraints or accounting identities, per-unit deviations may be equally or more convenient. More generally, mixed representations can be optimal in models combining both structures.

By making the role of coordinate choice explicit, this note aims to provide a clearer foundation for linear and quadratic approximations and to clarify common practices in applied macroeconomic analysis.

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